Laguerre group G_6' of circle transformations but only a certain subgroup G_3' of it. These remarks show that the equilong theory is entirely different in this respect from the conformal theory, where conformal symmetry with respect to a circle is Moebius inversion and hence generates the entire mixed six-parameter Moebius group G_6'.

4. *Other Distinctions between the Two Theories.* In the case of conformal (Schwarzian) symmetry S, it is essential that the base curve C shall be analytic. (This is used in the theory of analytic prolongation.) But in the case of our new equilong symmetry S^*, the base curve C may be any curve with continuously turning tangent (and even further generalization is possible).

If we take a *horn angle* and *bisect* it in the *conformal manner,* the intrinsic quantities curvature and the first and second derivatives with respect to arc length take exactly the average values for the middle curve; this is not true of the third and higher derivatives.[2]

If we bisect the horn angle in the *equilong manner,* then the appropriate intrinsic quantities are radius of curvature and derivatives with respect to inclination; and we find that these take *average values for all orders.*[1] Thus there are many analogies and many distinctions.

[1] Kasner, "Conformal and Equilong Symmetries," *Science,* **83,** 480 (1936).

[2] Kasner, "Geometry of Conformal Symmetry (Schwarzian Reflection), *Ann. Math.,* **38,** 873–879 (1937); Comenetz, "Conformal Geometry on a Surface," *Ibid.,* **39,** 863–871 (1938).

[3] Kasner, "Infinite Groups Generated by Conformal Transformations of Period Two (Involutions and Symmetries)," *Am. Jour. Math.,* **38,** 177–184 (1916).

PARTIALLY ORDERED SETS AND TOPOLOGY

BY ARTHUR N. MILGRAM

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME

Communicated March 12, 1940

1. *Separating Systems.*—In the following we obtain the Brouwer reduction theorem and the Borel covering theorems by applying theorems about partially ordered sets to systems of closed subsets of a topological space. Moreover, we formulate conditions on a partially ordered set P necessary and sufficient to guarantee the existence of a topological space having a basis of closed sets isomorphic to P. In both problems we use the notion of separating system.

Let $\mathfrak{T}$ be a collection of lower sections[1] of P. We call $\mathfrak{T}$ a separating system (strong separating system) if for each x and y of P such that $x < y$ (y not $\leq x$) there is a lower section L in the collection $\mathfrak{T}$ such that x belongs to L, and y does not belong to L. In an analogous way a collection of upper sections may be defined to be a separating system.

Many properties of partially ordered sets are consequences of the existence of a separating system whose power is sufficiently small.

2. *Extremal Sequences and the Reduction Theorem.*—Let P be a partially ordered set and $\mathfrak{L}$ a separating system of power M. For each element x of P either there is an element x' of P such that $x' \leq x$ and no element of P is $<$ than x', or there is a monotonically decreasing sequence $x_1 = x$, $x_2, \ldots, x_\alpha, \ldots$ defined for all $\alpha < \kappa$ where κ is an ordinal number not exceeding the first ordinal of power M. Such a sequence is called a minimal sequence starting with x.

We shall call P *lower inductive of power* M if, whenever $x_1, x_2, \ldots.$ is a monotonically decreasing sequence of elements of P of power not exceeding M, there exists an element x of P such that $x \leq x_\alpha$ for each x_α of the sequence. *If P is lower inductive of power M, and has a separating system of power at most M, then for each element x of P there exists at least one element $x' \leq x$ such that no other element of P is $< x'$.*

One consequence of this result is the Brouwer reduction theorem, since for a topological space having a denumerable basis, any partially ordered set of closed subsets has a denumerable separating system,[2] and the assumption that such a partially ordered set P is lower inductive of denumerable power implies that each of the sets belonging to P contains a smallest set belonging to P.

3. *The Zero Element in Subsets of a Partially Ordered Set P and the Covering Theorems.*—By the zero element of P is meant an element which is in the relation $<$ to every other element of P. If P contains no zero we may always adjoin one and call $P + 0$ the enlarged partially ordered set. From the results of section 2 we get: *If P is a partially ordered set with a denumerable separating system which contains a zero, and if P' is a subset of P which is lower inductive of denumerable power having the additional property that whenever $x \neq 0$ is an element of P' there is a $y < x$ in P', then 0 is an element of P'.*

This theorem yields the Borel covering theorem for completely separable spaces, i.e., that in a space S with a denumerable basis, from any covering by open sets it is possible to extract a denumerable number of open sets which alone cover S. Take P to be the set of all closed subsets of S, and P' to be the set of closed subsets of S which are the complementary sets to the sum of an at most denumerable number of open sets belonging to the covering. The conditions of the theorem hold, and the conclusion means that the vacuous set is the complement of a denumerable number of the open sets in the covering.[3]

Let P have a denumerable strong separating system, be lower inductive of denumberable power and have no zero element. Let P' be a subset of $P + 0$

such that 1) *if* $x \epsilon P$ *there exists an* $x' \epsilon P'$ *such that* x *not* $\leq x'$, *and* 2) *if* x, y *are elements of* P', *there exists a* $z \epsilon P'$ *such that* $z \leq x$ *and* $z \leq y$. *Under these conditions zero is an element of* P'. This theorem yields the Heine-Borel covering theorem for compact completely separable spaces. In the application the lower inductiveness of P is a consequence of the Cantor product theorem.

4. *Topological Spaces.*—Let P be a partially ordered set with a unit element having a strong separating system $\mathfrak{U}$ of upper sections. We assume 1) for any two distinct members of $\mathfrak{U}$, neither is a subset of the other, and 2) for each pair of elements, x, y of P, and each U of $\mathfrak{U}$ whenever neither x nor y is an element of U, there is a z in P such that $x \leq z$, $y \leq z$, and z is not an element of U.

We now define a topological space T as follows: The points of T are the members of $\mathfrak{U}$. A basis of closed sets of T is formed by the point sets T_x, where x is an element of P, and T_x the set of upper sections in $\mathfrak{U}$ containing x. If M is a subset of T we define the closure $\overline{M}$ to be the product of all sets T_x which contain M. It may readily be verified that a) if p is a point, then $\overline{p} = p$, b) $\overline{M + N} = \overline{M} + \overline{N}$, c) $\overline{\overline{M}} = \overline{M}$, d) the closure of the vacuous set is the vacuous set. The basis of closed sets T_x is in one-to-one correspondence with P preserving the order relations of $<$ in P and inclusion in T.

[1] A subset L of a partially ordered set P is called a lower section if whenever x is an element of L, $y < x$ implies that y is also an element of L. In an analogous way upper sections are defined.

[2] If the topological space S has a basis of open sets $\{0\}$ of power M, and if P is any partially ordered set of some (not necessarily all) closed subsets of S, then P has a strong separating system of power at most M. We obtain one if with each open set 0 of $\{0\}$ we associate the set U_0 of all closed sets which are elements of P and have a non-vacuous intersection with 0. It should be noted that U_0 is an upper section but is not an ideal even in the case that P consists of all closed subsets of S.

[3] The arguments about partially ordered sets with denumerable separating systems do not require the theory of transfinite numbers. The covering theorems for higher powers follow from general theorems about partially ordered sets with separating systems of higher powers.